2.2. Systems of Linear Equations

An $m \times n$ *system of linear equations* in variables $x_1, x_2, \ldots, x_n$ is a list of m equations of the form

$$\begin{cases} a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + & \cdots & + & a_{2n}x_n & = & b_2 \\ & \vdots & & & & & & & \vdots \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \cdots & + & a_{mn}x_n & = & b_m \end{cases}$$

Any point $(x_1, x_2, \ldots, x_n)$ which satisfies all the equations in the system is called a *solution* of the system.

We can represent such a system as $A\vec{\mathbf{x}} = \vec{\mathbf{b}}$:

$$\underbrace{\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}}_{\vec{\mathbf{x}}} = \underbrace{\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}}_{\vec{\mathbf{b}}}$$

Or, more succintly, we can write the *augmented matrix* $(A \mid \vec{\mathbf{b}})$:

$$\left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right]$$

Write the system of linear equations

$$\left\{ \begin{array}{rcrcrcr} 3x & + & y & - & 3z & = & 2 \\ x & + & y & - & z & = & 0 \\ x & & & - & z & = & 1 \end{array} \right\}$$

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$$\left(\begin{array}{ccc|c} 3 & 1 & -3 & 2 \\ 1 & 1 & -1 & 0 \\ 1 & 0 & -1 & 1 \end{array}\right)$$

2.3. Elementary Row Operations

Let $A \in M_{m,n}(\mathbb{R})$. The following operations are called **elementary row operations** (EROs) on the matrix A:

1. multiplying a row of A by a nonzero scalar. (If the i^{th} row of A is replaced by α times itself, the notation will be $\alpha R_i \to R_i$.)
2. interchanging two rows of A. (If the i^{th} and j^{th} rows of A are interchanged, the notation is $R_i \leftrightarrow R_j$.)
3. adding a scalar multiple of one row to another. (If row i of A is replaced by itself plus α times row j of A, the notation is $R_i + \alpha R_j \to R_i$.)

If $B \in M_{m,n}(\mathbb{R})$ is the result of applying a sequence of EROs to a given matrix $A \in M_{m,n}(\mathbb{R})$, then B is said to be **row equivalent** to A.

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Theorem 2.21: Let $M = (A|\vec{\mathbf{b}})$ be an augmented matrix corresponding to a linear system of equations $A\vec{\mathbf{x}} = \vec{\mathbf{b}}$, and let e be an ERO. Then the solution set of the linear system corresponding to the augmented matrix $e(M)$ is identical to the solution set of the linear system corresponding to M.

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Corollary 2.22: Linear systems that have row equivalent augmented matrices have identical solution spaces.